ON THE OSCILLATORY AND ROTATIONAL RESONANT MOTIONS

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In this paper we investigate the rotational and oscillatory solutions of perturbed, essentially nonlinear systems with several degrees of freedom. Using the method of small parameter we construct steady resonant solutions and apply the first Liapunov method to derive the sufficient conditions of their asymptotic stability. An example from nonlinear mechanics is solved to illustrate the proposed method. Analogous results were obtained earlier for the particular case of almost conservative systems with one degree of freedom. The system investigated in this paper represents a generalization of Liapunov and similar systems.

1. Statement of the problem. Let us consider a real system with a small parameter

$$dx_i / dt = F_i(x_1, \ldots, x_n) + \varepsilon f_i(t, x_1, \ldots, x_n, \varepsilon) \qquad (i = 1, \ldots, n) \tag{1.1}$$

for which the following generating self-contained system

$$dx_i^\circ / dt = F_i(x_1^\circ, \ldots, x_n^\circ) \equiv F_{i0} \tag{1.2}$$

admits a stable, two-parameter family of rotational – oscillatory solutions of the type [1]

$$x_i^\circ = \delta_i (T_i / 2\pi)\, \omega(E)(t - t_0 + \tau) + \varphi_i(\omega(E)(t - t_0 + \tau), E)$$

$$\delta_i = 1 \quad (i \leqslant p) \qquad \delta_i = 0 \quad (i > p,\ p \leqslant n)$$

When the system is purely oscillatory, we have $p = 0$, i.e. $x_i^\circ = \phi_i$, where ϕ_i are 2π-periodic functions of the phase $\psi = \omega(E)(t - t_0 + \tau)$; T_i denote the constant periods of the functions F_i and f_i in rotating coordinates; $\omega = \omega(E)$ is the natural frequency; τ is the phase constant and E is the second parameter of the family.

We can obtain (1.1) in a more suitable form using the following transformation [2 and 3]

$$x_i = x_i^\circ(\psi, E) + \frac{1}{2} \sum_{k=3}^{n} [A_{ik}(\psi, E) h_k + \bar{A}_{ik}(\psi, E) \bar{h}_k] \tag{1.3}$$

Here (A_{ik}) is an $n \times (n - 2)$-matrix which is, generally speaking, complex (a bar denotes a complex conjugate). This matrix also appears in the following nonsingular substitution

$$y_i = \frac{\partial x_i^\circ}{\partial \tau} u_1 + \frac{\partial x_i^\circ}{\partial E} u_2 + \sum_{k=3}^{n} A_{ik}(\psi, E) u_k$$

which reduces the system of nonperturbed equations written in variational form

$$\frac{dy_i}{dt} = \sum_{k=1}^{n} \left(\frac{\partial F_i}{\partial x_k}\right)_0 y_k \qquad (i = 1, \ldots, n) \tag{1.4}$$

to a system with constant coefficients of the form

$$\frac{du_1}{dt} = \omega'(E)\,u_2, \qquad \frac{du_2}{dt} = 0, \qquad \frac{du_j}{dt} = \sum_{k=3}^{n} H_{jk}(E)\,u_k \qquad (j = 3, \ldots, n)$$

in which the roots of the characteristic Eq.

$$\Delta(\rho) = |H_{jk} - \delta_{jk}\rho| = 0 \qquad (j, k = 3, \ldots, n)$$

will also play the part of characteristic indices for the variational system (1.4), whose variations have negative real parts (the remaining two are equal to zero). As a result we obtain the following system (1.5)

$$\frac{dE}{dt} = f(t, E, \psi, h, \varepsilon), \quad \frac{d\psi}{dt} = \omega(E) + F(t, E, \psi, h, \varepsilon), \quad \frac{dh}{dt} = H(E)h + g(t, E, \psi, h, \varepsilon)$$

Here h is the $(n-2)$-dimensional vector and $H(E)$ is the $(n-2)\times(n-2)$-stable matrix, both of them are complex quantities. Functions f, ω, F and g are real for real E, ψ and ε and complex h, are θ-periodic in t and 2π-periodic in ψ, both periods being constant (θ denotes the period of f_t in t). Further, the following estimates hold for the functions, f, F and g when $|\varepsilon|$ and $|h|$ are sufficiently small:

$$|f|, \quad |F|, \quad |g| \leqslant \lambda_1|\varepsilon| + \lambda_2|h|^2 \qquad (\lambda_1, \lambda_2 < +\infty) \tag{1.6}$$

At $\varepsilon = 0$ and $h = 0$ these inequalities yield the following identities

$$(f, F, g) \equiv 0, \quad \frac{\partial^{r+s}(f, F, g)}{\partial\psi^r\,\partial F^s} \equiv 0, \quad \frac{\partial^{r+s+1}(f, F, g)}{\partial h\,\partial\psi^r\,\partial E^s} \equiv 0 \qquad (r, s \geqslant 1)$$

provided that f, F and g are differentiable the required number of times.

In this paper we develop a direct method of constructing steady resonant solutions of the system (1.5) for all $t \in [t_0, \infty)$. Unlike the existing averaging schemes [1 and 2] our method of small parameter enables us to follow the behavior of the perturbed system in the limit as $t \to \infty$. To put it more accurately, our scheme yields the sufficient conditions for the occurrence of the steady, resonant modes. When studying the Liapunov stability of these modes, we find that we are able to follow the development of other similar type modes at the initial time. This throws light on the importance of the study of the Liapunov stability of a perturbed motion. We note that the unperturbed motion is unstable and, that we have a critical case in which one group of solutions corresponds to a double characteristic index equal to zero.

2. Construction of the steady resonant solution of the system.

The solution will be a resonant one of the form m/l, if

$$\omega(E_0^*)/\nu = l/m \qquad (\nu = 2\pi/\theta)$$

where m and l are integers in some simple ratio. If the functions f, ω, F, H and g are analytic in some region

$$|\varepsilon| \leqslant \varepsilon_0, \quad |E - E_0^*| \leqslant \alpha, \quad |\mathrm{Im}\,\psi| \leqslant \beta, \quad |h| \leqslant \sigma$$

then the solution should be sought [4] in the form of series

$$E = E_0^* + \sum_{i=1}^{\infty} \varepsilon^i E_i, \qquad \psi = \frac{l}{m}\nu(t - t_0) + \tau + \sum_{i=1}^{\infty} \varepsilon^i \psi_i, \qquad h = \sum_{i=1}^{\infty} \varepsilon^i h_i \tag{2.1}$$

in which E_i, ψ_i and h_i ($i \geq 1$) are ($T = m\theta$)-periodic. Using the estimates (1.6) we find, that the functions

$$E = E_0^*, \qquad \psi = l/m\nu\,(t - t_0) + \tau, \quad h = 0$$

will be a solution of (1.5) when $\varepsilon = 0$. Inserting the series (2.1) into (1.5) and comparing the coefficients of like powers of ε, we obtain an infinite sequence of interrelated systems for E_i, ψ_i and h_i and in particular, the following system for the first increments

$$\frac{dE_1}{dt} = \left(\frac{\partial f}{\partial\varepsilon}\right)_0, \qquad \frac{d\psi_1}{dt} = \omega_0' E_1 + \left(\frac{\partial F}{\partial\varepsilon}\right)_0, \qquad \frac{dh_1}{dt} = H_0 h_1 + \left(\frac{\partial g}{\partial\varepsilon}\right)_0$$

Here the subscript 0 at the relevant expressions in brackets, mean that they are taken for the generation solution and for $\varepsilon = 0$. The first elementary equation yields

$$E_1 = \int_{t_0}^{t} \left(\frac{\partial f}{\partial \varepsilon}\right)_0 dt_1 + A_1 \qquad (A_1 = \text{const})$$

From the above argument it follows that E_1 will be periodic function, if the phase constant satisfies

$$P(\tau) = \int_0^T \left(\frac{\partial f}{\partial \varepsilon}\right)_0 dt = 0 \qquad (\operatorname{Im} \tau^* = 0) \tag{2.2}$$

This condition is necessary and sufficient under the constraints imposed on the function f and other quantities. We shall call (2.2) the condition of phase equilibrium. Later we shall see that the condition of periodicity together with other similar conditions have the decisive role in our investigation, since they will be used for elimination the secular terms.

We find the function ψ_1 in the similar manner

$$\psi_1 = \omega_0' A_1 (t - t_0) + \int_{t_0}^{t} \left[\omega_0' \int_{t_0}^{t_1} \left(\frac{\partial f}{\partial \varepsilon}\right)_0 dt_2 + \left(\frac{\partial F}{\partial \varepsilon}\right)_0\right] dt_1 + B_1$$

Condition of periodicity together with the condition that $\omega'(E_0^*) \neq 0$, yield the value of the constant A_1

$$A_1^* = -(\omega_0' T)^{-1} \int_0^T \left[\omega_0' \int_{t_0}^{t} \left(\frac{\partial f}{\partial \varepsilon}\right)_0 dt_1 + \left(\frac{\partial F}{\partial \varepsilon}\right)_0\right] dt$$

This defines the periodic functions E_1 and h_1 completely

$$E_1 = \int_{t_0}^{t} \left(\frac{\partial f}{\partial \varepsilon}\right)_0 dt_1 + A_1^*, \qquad h_1 = \int_{-\infty}^{t} e^{H_0 (t - t_1)} \left(\frac{\partial g}{\partial \varepsilon}\right)_0 dt_1$$

while ψ_1 is defined with accuracy of up to the constant B_1.

Equations for second increments yield

$$E_2 = B_1 \int_0^{t} \left(\frac{\partial^2 f}{\partial \varepsilon \, \partial \psi}\right)_0 dt_1 + \int_{t_0}^{t} S_2(t_1)\, dt_1 + A_2 \qquad (A_2 = \text{const})$$

where S_2 is a known periodic function

$$S_2(t) = \frac{1}{2}\left(\frac{\partial^2 f}{\partial \varepsilon^2}\right)_0 + \left(\frac{\partial^2 f}{\partial \varepsilon \, \partial E}\right)_0 E_1 + \left(\frac{\partial^2 f}{\partial \varepsilon \, \partial h}\right)_0 h_1 + \frac{1}{2}\left(\frac{\partial^2 f}{\partial h^2}\right)_0 h_1^2 +$$
$$+ \left(\frac{\partial^2 f}{\partial \varepsilon \, \partial \psi}\right)_0 \int_{t_0}^{t} \left[\omega_0' \int_{t_0}^{t_1} \left(\frac{\partial f}{\partial \varepsilon}\right)_0 dt_2 + \left(\frac{\partial F}{\partial \varepsilon}\right)_0 + \omega_0' A_1^*\right] dt_1$$

Condition of periodicity of E_2 yields B_1

$$B_1^* = -\left(\frac{\partial P}{\partial \tau^*}\right)^{-1} \int_0^T S_2(t)\, dt$$

provided that τ^* is a simple, real root of (2.2). This defines the periodic function ψ_1. Inserting the latter into the expression for E_2 and ψ_2 we find

$$A_2^* = -(\omega_0' T)^{-1} \int_0^T \left\{\omega_0' \int_{t_0}^{t} \left[B_1^* \left(\frac{\partial^2 f}{\partial \varepsilon \, \partial \psi}\right)_0 + S_2\right] dt_1 + \frac{1}{2}\omega_0'' E_1^2 + \right.$$
$$\left. + \frac{1}{2}\left(\frac{\partial^2 F}{\partial \varepsilon^2}\right)_0 + \left(\frac{\partial^2 F}{\partial \varepsilon \, \partial E}\right)_0 E_1 + \left(\frac{\partial^2 F}{\partial \varepsilon \, \partial \psi}\right)_0 \psi_1 + \left(\frac{\partial^2 F}{\partial \varepsilon \, \partial h}\right)_0 h_1 - \frac{1}{2}\left(\frac{\partial^2 F}{\partial h^2}\right)_0 h_1^2\right\} dt$$

Thus the periodic functions E_2 and h_2 can be fully determined from the second approximation system, while

$$\psi_2 = \int_{t_0}^{t} \left[\omega_0' E_2 + \frac{1}{2}\omega_0'' E_1^2 + \frac{1}{2}\left(\frac{\partial^2 F}{\partial \varepsilon^2}\right)_0 + \left(\frac{\partial^2 F}{\partial \varepsilon \partial E}\right)_0 E_1 + \right.$$
$$\left. + \left(\frac{\partial^2 F}{\partial \varepsilon \partial \psi}\right)_0 \psi_1 + \left(\frac{\partial^2 F}{\partial \varepsilon\, \partial h}\right)_0 h_1 + \frac{1}{2}\left(\frac{\partial^2 F}{\partial h^2}\right)_0 h_1^2 \right] dt_1 + B_2.$$

The constant of integration B_2 appearing in this expression is obtained from the condition of periodicity of E_3 etc.

We can find in this manner the corrections of any order and prove by induction that the method yields any required number of bounded periodic coefficients of the series (2.1). This means that we can obtain a resonant solution which will be unique within the domain of definition and analyticity of the system (1.5) up to any degree of accuracy in ε for all $t \in [t_0, \infty)$.

N o t e 2.1. A steady resonant solution of (1.5) can be constructed using consecutive approximations with the help of the following system

$$\frac{dx_i}{dt} = \left(\frac{\partial f}{\partial \varepsilon}\right)_0 + \varepsilon \left[\frac{1}{2}\left(\frac{\partial^2 f}{\partial \varepsilon^2}\right)_0 + \left(\frac{\partial^2 f}{\partial \varepsilon\, \partial E}\right)_0 x_{i-1} + \left(\frac{\partial^2 f}{\partial \varepsilon \partial \psi}\right)_0 y_{i-1} + \right.$$
$$\left. + \left(\frac{\partial^2 f}{\partial \varepsilon\, \partial h}\right)_0 z_{i-1} + \frac{1}{2}\left(\frac{\partial^2 f}{\partial h^2}\right)_0 z_{i-1}^2 + X(t,\ x_{i-1}, y_{i-1}, z_{i-1}, \varepsilon) \right]$$

$$\frac{dy_i}{dt} = \omega_0' x_i + \left(\frac{\partial F}{\partial \varepsilon}\right)_0 + \varepsilon \left[\frac{1}{2}\omega_0'' x_{i-1}^2 + \frac{1}{2}\left(\frac{\partial^2 F}{\partial \varepsilon^2}\right)_0 + \left(\frac{\partial^2 F}{\partial \varepsilon\, \partial E}\right)_0 x_{i-1} + \right.$$
$$\left. + \left(\frac{\partial^2 F}{\partial \varepsilon \partial \psi}\right)_0 y_{i-1} + \left(\frac{\partial^2 F}{\partial \varepsilon\, \partial h}\right)_0 z_{i-1} + \frac{1}{2}\left(\frac{\partial^2 F}{\partial h^2}\right)_0 z_{i-1}^2 + Y(t, x_{i-1}, y_{i-1}, z_{i-1}, \varepsilon) \right]$$

$$\frac{dz_i}{dt} = H_0 z_i + \left(\frac{\partial g}{\partial \varepsilon}\right)_0 + \varepsilon \left[\frac{1}{2}\left(\frac{\partial^2 g}{\partial \varepsilon^2}\right)_0 + \left(\frac{\partial^2 g}{\partial \varepsilon\, \partial E}\right)_0 x_{i-1} + \left(\frac{\partial^2 g}{\partial \varepsilon \partial \psi}\right)_0 y_{i-1} + \left(\frac{\partial^2 g}{\partial \varepsilon\, \partial h}\right)_0 z_{i-1} + \right.$$
$$\left. + \frac{1}{2}\left(\frac{\partial^2 g}{\partial h^2}\right)_0 z_{i-1}^2 + \left(\frac{\partial H}{\partial E}\right)_0 x_{i-1} z_{i-1} + Z(t, x_{i-1}, y_{i-1}, z_{i-1}, \varepsilon) \right]$$

Here X, Y and Z are known, sufficiently smooth functions. Proof of the convergence of consecutive approximations i.e. of the convergence of the functions x_i, y_i and z_i to T-periodic functions appearing in the following substitution

$$E = E_0^* + \varepsilon x, \qquad \psi = (l\ /\ m)\, \nu\, (t - t_0) + \tau + \varepsilon y, \qquad h = \varepsilon z \tag{2.3}$$

is given in [4], therefore we consider the use of the proposed system justified. It should be noted that the method of consecutive approximations can be applied to systems of the type (1.5), if the functions f, ω, F and g posses first and second order partial derivatives with respect to ε, E, ψ and h satisfying, together with dH/dE the Lipshits conditions with t-independent constants in some region

$$\varepsilon \in [0, \varepsilon_0], \quad -\alpha \leqslant E - E_0^* \leqslant \alpha, \; \psi \in (-\infty, \infty), \; |h| \leqslant \sigma$$

The result obtained can be formulated briefly as follows:

T h e o r e m 2.1. If

1) functions f, ω, F, H and g are sufficiently smooth and satisfy the conditions listed in Section 1;

2) equations(2.2) has a real root τ^* and

3) the inequality

$$\omega'(E_0^*)\ \partial P\ /\ \partial \tau^* \neq 0$$

holds, then provided that $|\varepsilon|$ is sufficiently small, the perturbed system (1.5) has a unique steady resonant solution belonging to the domain of definition and smoothness of the functions f, ω, F, H and g. When $\varepsilon = 0$, this solution is

$$E = E_0^* = \text{const}, \qquad \psi = (l\ /\ m)\, \nu\, (t - t_0) + \tau^*, \qquad h = 0$$

N o t e 2.2. When we say "the uniqueness of the solution" we mean, that a single solution of the type (2.3) corresponds to a fixed set of values of m, l, E_0^* and τ^*. It can be easily be shown that Eq. (2.2) admits, on the segment of length 2π, andeven number of real roots τ^*.

Critical cases are possible, when the condition (3) of the Theorem does not hold.

N o t e 2.3. Let τ^* be a real, r-tuple ($r \geq 2$) root of (2.2), but let $\omega_0' \neq 0$. In this case the uniqueness of the solution as defined above, may be violated. In general, we can represent the steady resonant solution in the form of a series in fractional powers of a small parameter. Integration constants of the type B_i can be found from nonlinear algebraic equations. Investigation of the generalized case is difficult and demands the use of subtle and involved results of the theory of implicit analytic functions. The pattern of splitting of the integral curves appears, in this case, to be very complex.

N o t e 2.4. The case when (2.2) is satisfied identically for some m and l, is fairly often met in practice. We then speak of higher order motions. Malkin in [4] indicated the possibility of occurrence of such cases for the oscillatory nonlinear systems, while investigating the periodic solution by Poincare's method. He also investigated an analogous particular case for a quasilinear resonant system. Periodic resonant motions of the first, second and third order are obtained in [5] for a nonlinear analytic system with one degree of freedom and their Liapunov stability is investigated. Analogous rotational problem was dealt with in [6].

We should note that this critical case is of considerable theoretical and practical interest when applied to the general system (1.1) and should be studied in detail.

N o t e 2.5. Critical cases occurring when $\omega_0' = 0$ are of practical interest, provided that ω is independent of E, i.e. provided that the system is quasilinear. Real constants E_0^* and τ^* defining the steady mode can then be obtained from

$$P(E_0, \tau) \equiv \int_0^T \left(\frac{\partial f}{\partial \varepsilon}\right)_0 dt = 0, \qquad Q(E_0, \tau) \equiv \int_0^T \left(\frac{\partial F}{\partial \varepsilon}\right)_0 dt = 0$$

while the condition (3) of Theorem 2.1 assumes the form

$$\partial(P, Q) / \partial(E_0^*, \tau^*) \neq 0 \tag{2.4}$$

If, on the other hand, ω is dependent on E but $\omega_0' \neq 0$, then the uniqueness of may be violated for some specified set of m and l. We call such a case an exceptional one when dealing with nonlinear motions. Obviously, we can always achieve the condition $\omega_0' \neq 0$ by varying m and l.

3. Investigation of stability of the perturbed resonant solution.

We shall use the substitution

$$E = E(t, \varepsilon) + U, \qquad \psi = \psi(t, \varepsilon) + V, \qquad h = h(t, \varepsilon) + W$$

to construct the following variational equations

$$\frac{dU}{dt} = \frac{\partial f}{\partial E} U + \frac{\partial f}{\partial \psi} V + \frac{\partial f}{\partial h} W + f_1(t, U, V, W, \varepsilon)$$

$$\frac{dV}{dt} = \left(\omega' + \frac{\partial F}{\partial E}\right) U + \frac{\partial F}{\partial \psi} V + \frac{\partial F}{\partial h} W + F_1(t, U, V, W, \varepsilon)$$

$$\frac{dW}{dt} = \left(H'z + \frac{\partial g}{\partial E}\right) U + \frac{\partial g}{\partial \psi} V + \left(H + \frac{\partial g}{\partial h}\right) W + g_1(t, U, V, W, \varepsilon)$$

Here the functions f_1, F_1 and g_1 are periodic in t, and the first terms of their expansions in U, V and W are quadratic. The well-known Liapunov's theorem [4] implies that it is sufficient to investigate the stability of the stagnation point of the linear approximating system.

When $\varepsilon = 0$, we see that $(n - 2)$ characteristic indices of the variational system have negative real parts, while two remaining indices have both, real and imaginary parts, equal to zero. These two indices have a single corresponding group of solutions. In this case the expansion of the critical characteristic indices will be in the powers of $\delta = \sqrt{\varepsilon}$. One of the solutions of the variational system has the form

$$U = u\exp\gamma t, \quad V = v\exp\gamma t, \quad W = w\exp\gamma t$$

where γ is the critical characteristic index, while u, v and w are periodic functions of t. Moreover,

$$\gamma = \sum_{i=1}^{\infty} \delta^i \gamma_i, \quad u = \sum_{i=0}^{\infty} \delta^i u_i, \quad v = \sum_{i=0}^{\infty} \delta^i v_i, \quad w = \sum_{i=0}^{\infty} \delta^i w_i$$

i.e. the functions u_i, v_i and w_i $(i \geq 0)$ should also be T-periodic. Taking this into account we obtain, from the conditions of periodicity of u_0, v_0, w_0, u_1, v_1, w_1 and u_2, the following relation for γ_1

$$\gamma_1^2 = \omega'(E_0^*)\ \partial P / \partial\tau^*$$

Thus, when $\gamma_1^2 > 0$, the perturbed solution is unstable for $t \gtrless t_0$; if, on the other hand, $\gamma_1^2 < 0$, then its stability depends on the sign of γ_2 which can be found from the conditions of periodicity of the functions v_1, w_2 and u_3, and is

$$\gamma_2 = \frac{1}{2T} \int_0^T \left[\left(\frac{\partial^2 f}{\partial\varepsilon\, \partial E} \right)_0 + \left(\frac{\partial^2 F}{\partial\varepsilon\, \partial\psi} \right)_0 \right] dt$$

As a result, we have the following expression for both critical characteristic indices

$$\gamma = \pm\delta\,(\omega_0' \partial P / \partial\tau^*)^{1/2} + \delta^2\gamma_2 + O(\delta^3)$$

which yields, at sufficiently small $\varepsilon > 0$, the following theorem.

T h e o r e m 3.1. The constructed perturbed resonant solution (2.3) is Liapunov stable as well as asymptotically stable for $t \geq t_0$, provided that

$$\omega'(E_0^*) \frac{\partial P}{\partial\tau^*} < 0, \qquad \int_0^T \left[\left(\frac{\partial^2 f}{\partial\varepsilon\, \partial E} \right)_0 + \left(\frac{\partial^2 F}{\partial\varepsilon\, \partial\psi} \right)_0 \right] dt < 0$$

and unstable otherwise.

N o t e 3.1. Theorem 2.1 excludes the case $\gamma_1 = 0$, i.e. the first condition is the necessary one. If $\gamma_2 = 0$, then higher powers of δ must be taken into account (computation of γ_3 etc.) to obtain the sufficient conditions of stability.

N o t e 3.2. When $\omega = \text{const}$, the resonant solution will be asymptotically stable if the eigenvalues of the matrix (2.4) have negative real parts (see Note 2.5).

In conclusion we shall consider a specific example taken from mechanics.

4. Example. We consider a mechanical model representing a system with two degrees of freedom in the gravity field (Fig. 1). We assume that the model is fixed to a rigid support at two points, and that the forces acting at these points on the plane annulus consist of a recurrent elastic moment and a frictional moment proportional to the angular velocity By constructing a Lagrangian for this system with the perturbation forces taken into account and performing the relevant differentiation, we can obtain the system

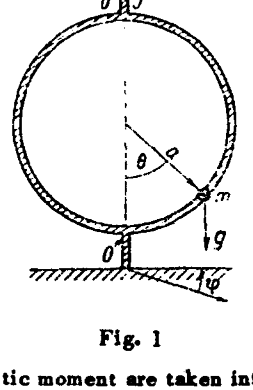

Fig. 1

$$ma^2\theta'' - ma^2 \sin\theta \cos\theta\varphi'^2 + mga \sin\theta = -\alpha_1\theta' + f_1(\nu t)$$

$$I\varphi'' + ma^2 \sin^2\theta\varphi'' + 2ma^2 \sin\theta \cos\theta\theta'\varphi' + k_1^2\varphi = -$$
$$- \alpha_1 \sin^2\theta\varphi' - \lambda_1\varphi' - z_1(\varphi) + g_1(\nu t)$$

Here I denotes the moment of inertia of the ring relative to the $O'OO''$-axis, α_1 is the coefficient of viscous friction between the ball and the outer medium, f_1 and g_1 are the external periodic moments and z_1 is a function in which the nonlinear effects of the elastic moment are taken into account. Using the notation

$$\frac{ma^2}{I} = \varepsilon, \quad \frac{\alpha_1}{ma^2} = \varepsilon\alpha, \quad \frac{f_1(\nu t)}{ma^2} = \varepsilon f(\nu t), \quad \frac{k_1^2}{I} = k^2$$

$$\frac{z_1(\varphi)}{I} = \varepsilon z(\varphi), \quad \frac{\lambda_1}{I} = \lambda, \quad \frac{\alpha_1}{I} = \varepsilon^2\alpha, \quad \frac{g_1(\nu t)}{I} = \varepsilon g(\nu t)$$

we can obtain a system of the type (1.1)

$$\theta^{\cdot\cdot} - \sin\theta\cos\theta\varphi^{\cdot 2} + (g/a)\sin\theta = \varepsilon\,[f(\nu t) - \alpha\theta^{\cdot}]$$

$$\varphi^{\cdot\cdot} + \lambda\varphi^{\cdot} + k^2\varphi = \varepsilon\,(1 + \varepsilon\sin^2\theta)^{-1}[g(\nu t) + k^2\varphi\sin^2\theta + (\lambda - \alpha)\varphi^{\cdot}\sin^2\theta - 2\theta^{\cdot}\varphi^{\cdot}\sin\theta\cos\theta - z(\varphi)] \tag{4.1}$$

We shall, for definiteness, consider the case

$$f(\nu t) \equiv f_0\sin\nu t, \qquad g(\nu t) \equiv g_0\sin(\nu t + \delta), \qquad z(\varphi) \equiv \sigma\varphi^3$$

When $\varepsilon = 0$, the system (4.1) admits a two-parameter family of periodic

$$\theta_0 = 2\arcsin(\gamma_1\,\mathrm{sn}\,[2\sqrt{g/a}\,(t+\tau),\ \gamma_1]), \qquad \varphi_0 = 0$$

$$(\gamma_1 = \sqrt{aE_0/2g}, \quad T_0(E_0) = 2\sqrt{a/g}\,K(\gamma_1), \quad \gamma_1 < 1)$$

or rotational-oscillatory

$$\theta_0 = 2\,\mathrm{am}\,[\sqrt{E_0/2}\,(t+\tau),\ \gamma_2] = \omega(E_0)(t+\tau) + 4\sum_{j=1}^{\infty}\frac{1}{j}\frac{q^j}{1+q^{2j}}\sin j\omega(E_0)(t+\tau)$$

$$\varphi_0 = 0 \qquad (\gamma_2 = 1/\gamma_1 < 1, \quad T_0(E_0) = 2\sqrt{2/E_0}\,K(\gamma_2), \quad q = \exp - \pi K'/K)$$

solutions. Here K denotes a complete elliptic integral of the first kind taken over the corresponding moduli, while E_0 and τ are constants of integration. In the following we shall limit ourselves to the rotational-oscillatory solutions. Using the substitution

$$\theta = \theta_0(\psi, E), \qquad \theta^{\cdot} = \theta^{\cdot}_0(\psi, E), \qquad h_1 = \varphi, \qquad h_2 = \varphi^{\cdot}$$

we can obtain the following system of the type (1.5):

$$dE/dt = \varepsilon\theta_0^{\cdot}(f_0\sin\nu t - \alpha\theta_0^{\cdot}) + \theta_0^{\cdot}\sin\theta_0\cos\theta_0\,h_2^2$$

$$d\psi/dt = \sqrt{E/2}\,\pi/K(\gamma_2) + \theta_0^{\cdot}\,[\varepsilon(f_0\sin\nu t - \alpha\theta_0^{\cdot}) + \sin\theta_0\cos\theta_0 h_2^2]\,\times$$

$$\times\int_0^{\theta_0}\{\omega_0'[2E - 2g/a\,(1-\cos x)]^{-1/2} - \omega_0[2E - 2g/a\,(1-\cos x)]^{-3/2}\}\,dx$$

$$dh_1/dt = h_2, \quad dh_2/dt = -k^2h_1 - \lambda h_2 + \varepsilon\,(1 + \varepsilon\sin^2\theta_0)^{-1}\,\times$$

$$\times\,[g_0\sin(\nu t + \delta) + (k^2h_1 + (\lambda - \alpha)h_2)\sin^2\theta_0 - h_2\theta_0^{\cdot}\sin 2\theta_0 - \sigma h_1^3]$$

and apply to it the method developed in Sections 2 and 3.

We can, however, investigate the system (4.1) directly. Substituting into it the series

$$\theta = \theta_0(\psi_0, E_0) + \varepsilon\theta_1(t) + \ldots, \qquad \varphi = \varepsilon\varphi_1(t) + \varepsilon^2\varphi_2(t) + \ldots$$

and comparing the coefficients of like powers of ε we obtain, in particular,

$$\theta_1^{\cdot\cdot} + (g/a)\theta_1\cos\theta_0(\psi_0, E_0) = f_0\sin\nu t - \alpha\theta_0^{\cdot}(\psi_0, E_0)$$

$$\varphi_1^{\cdot\cdot} + \lambda\varphi_1^{\cdot} + k^2\varphi_1 = g_0\sin(\nu t + \delta)$$

Periodic solution of this linear system can be obtained in its explicit form using the method of variation of the integration constants

$$\theta_1 = M_1\theta_0^{\cdot} + \frac{1}{\Delta}\left\{\theta_0^{\cdot}\int_0^t\left[\int_0^{t_1}(f_0\sin\nu t_2 - \alpha\theta_0^{\cdot})\,\theta_0^{\cdot}dt_2 - \omega_0\frac{\partial\theta_0}{\partial\omega_0}(f_0\sin\nu t_1 - \alpha\theta_0^{\cdot}) - N_1\right]dt_1 + \right.$$

$$\left. + \,\omega_0\frac{\partial\theta_0}{\partial\omega_0}\left[\int_0^t(f_0\sin\nu t_1 - \alpha\theta_0^{\cdot})\,\theta_0^{\cdot}dt_1 - N_1\right]\right\} \equiv M_1\theta_0^{\cdot} + \theta_1^{*}$$

$$N_1 = \frac{1}{T}\int_0^T\left[\int_0^t(f_0\sin\nu t_1 - \alpha\theta_0^{\cdot})\,\theta_0^{\cdot}dt_1 - \omega_0\frac{\partial\theta_0}{\partial\omega_0}(f_0\sin\nu t - \alpha\theta_0^{\cdot})\right]dt$$

$$\varphi_1 = [g_0(k^2 - \nu^2)\sin(\nu t + \delta) - g_0\lambda\nu\cos(\nu t + \delta)]\,/\,[(k^2 - \nu^2)^2 + (\lambda\nu)^2]$$

Here M_1 and N_1 are constants of integration, while $\Delta = \Delta(0)$ is a Wronskian for the Eq.

$$x^{\cdot\cdot} + (g / a)\cos^2\theta_0 x = 0 \qquad (x_1 = \theta^{\cdot}_0, \quad x_2 = \theta^{\cdot}_0 t + \omega_0 \partial\theta_0 / \partial\omega_0)$$

where the brackets contain its basic system of solutions. For simplicity we shall only consider the resonance of the form $m : 1$. Then the equation of phase equilibrium can be written as

$$P(\tau) \equiv -\frac{4\pi m}{\nu}\frac{q^m}{1+q^{2m}}\sin\nu\tau - \frac{8\alpha\nu}{\pi m}\frac{G(\gamma_2)}{K(\gamma_2)} = 0$$

where G denotes a complete elliptic integral of the second kind. This equation admits the following real roots

$$\beta = 2\nu^2\alpha G(\gamma_2)\,(1 + q^{2m}) / \pi^2 m^2 K(\gamma_2) q^m \leqslant 1$$

provided that the inequality

$$\tau_1 = -(1/\nu)\arcsin\beta, \qquad \tau_2 = (1/\nu)\,(\pi + \arcsin\beta) \pmod{2\pi}$$

holds. If $\beta < 1$ (the case $\beta = 1$ is a critical one) we have $\partial P/\partial\tau^* \neq 0$ and by Theorem 2.1 there exists a solution of the perturbed system provided that ε is sufficiently small. In particular, Expressions

$$\theta = \theta_0 + \varepsilon\,(M_1^*\theta_0^{\cdot} + \theta_1^*) + O(\varepsilon^2), \qquad \varphi = \varepsilon\varphi_1(t) + \varepsilon^2\varphi_2(t) + O(\varepsilon^3)\,]$$

hold for $t \in [0, \infty)$. Here we use the following notation

$$M_1^* = \left(\frac{\partial P}{\partial\tau^*}\right)^{-1}\int_0^T \left(2\alpha\theta_1^{\cdot *} - \theta_1^{\cdot *} f_0 \sin\nu t - \frac{1}{2}\sin 2\theta_0\varphi_1^{\cdot 2}\right)\theta_0^{\cdot}\,dt$$

$$\varphi_2(t) = \frac{1}{p_1 - p_2}\int_{-\infty}^{t} [e^{p_1(t-t_1)} - e^{p_2(t-t_1)}]\,\{-\theta_0^{\cdot}\varphi_1^{\cdot}\sin 2\theta_0 +$$

$$+ \sin^2\theta_0\,[(\lambda - \alpha)\varphi_1^{\cdot} + k^2\varphi_1 - g_0\sin(\nu t_1 + \delta)]\}\,dt_1$$

$$(p_{1,2} = -\lambda/2 \pm \sqrt{\lambda^2/4 - k^2}, \qquad \partial P/\partial\tau^* = \mp 4\pi m q^m \sqrt{1-\beta^2}/(1 + q^{2m}))$$

Moreover, by Theorem 3.1 we can establish that the perturbed solution is asymptotically stable for $\tau^* = \tau_1$, provided that $\alpha > 0$.

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